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A theory of continua with projective microstructure as a model for large trusses

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SUMMARY

A large truss is approximated by a continuum admitting projective transformations as microscopic deformations. A suitable set of parameters describing the microscopic deformation is extracted, and its spatial compatibility is investigated. Equations of equilibrium are derived by means of a variational principle. They define a special class of materials of grade 2. A complete solution is expressed in terms of potential functions in the three-dimensional isotropic field, which corresponds to a truss of random connection. Stresses around cylindrical and spherical cavities are analyzed to see the singular effects near the boundary surface.

1. Introduction

Static and dynamic responses of a structure are analyzed by the method of structural analysis. If the structure is large, however, a large number of algebraic equations must be solved. Numerical solutions can be obtained by the use of electronic computers, but calculations become expensive, when the number of equations is large. Hence a suitable technique for approximation is desired. For this purpose continuous approximation has been frequently used; elastic plates and shells have long been used as models for lattice plates and shells. Recently, Cosserat or micropolar continua [1] have been used to describe rotations of joints of a grid framework as a field variable [2–6].

In general continuous approximation has one disadvantage: analytical solutions of the differential equations approximating the system are very difficult to obtain. In this respect the direct numerical analysis of the original discrete system is much more effective. However, we must avoid always resorting to numerical calculations, which require much labor and money, without any prospective insight. In contrast to the direct analysis the continuous approach makes theoretical forecast easy; if we know the differential equations approximating the system, then we can read characteristics of the equivalent field. We can foresee various effects, whose actual magnitude is then determined by the direct analysis. Since this is the real merit of the continuous approximation, the equivalent continuum must be defined in such a way that it well represents the internal mechanism of the corresponding discrete system. The Cosserat or micropolar continuum models for grid frameworks are good examples. In this paper we shall show that elastic approximation of a truss is insufficient to describe the internal complexity and define a new continuum which admits projective transformations as microscopic deformations.

The classical continuum is an ideal material, any small portion of which has the same mechanical properties possessed as a whole. In real materials, however, there is a lower limit of scale, below which mechanical characteristics differ from those macroscopically observed. In the continuous description of such materials the microscopic state of a limit element is expressed by additional field variables. Various kinds of additional variables can be attached in the form of vectors and tensors so that they can well represent the microscopic structure of the material [7–11]. Generalizing these results, we can say that the microscopic limit element corresponds to a topological space on which operates some transitive transformation group, which is called the admitted microscopic deformations. If the transformation group is taken to be that of rotations, the continuum is that of Cosserat [7, 8], and if the group is that of affine transformations, the continuum belongs to Mindlin's materials of microstructure [10], or Eringen's micromorphic materials [11]. In this paper we shall take the projective transformation group as the admitted microscopic deformations to define a continuum model for a truss, for a truss has the intrinsic projective property that a line element is mapped on a line element. We extract a set of parameters specifying the microscopic deformations, or in other words the coordinates of the group, and investigate its spatial compatibility. Then the equations of equilibrium are derived by means of a variational principle. The boundary conditions are obtained at the same time. These equations define a special class of materials of grade 2 [12]. Then, following the procedure of Mindlin and Tiersten [13], we exhibit a complete solution in the three-dimensional space. The constitutive equations are taken to be isotropic for simplicity. The field then represents a truss of random internal connection. The solution is expressed in terms of four sets of potential functions, two of which are those of the elastic field. Finally, stresses around cylindrical and spherical cavities are analyzed to see the singular effects near the boundary surface.

2. Projective deformations of a truss unit

Consider the truss beam of Fig. 1. It is usually approximated by an elastic beam with the equivalent bending stiffness

$$\tilde{E}\tilde{I} = \left(\frac{h}{2}\right)^2 EA + \left(\frac{h}{2}\right)^2 EA = \tfrac{1}{2}EAh^2, \tag{2.1}$$

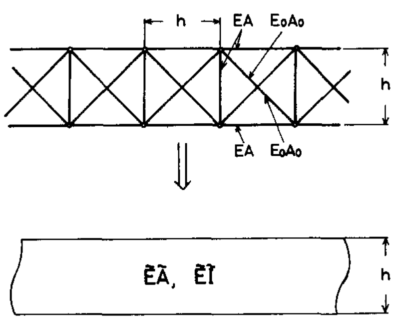

Figure 1. Continuous approximation of a truss beam.

where EA is the longitudinal stiffness of the upper and the lower members. The longitudinal stiffness E_0A_0 of the diagonal members plays no role in bending. This approximation is actually in fairly good agreement with exact solutions. Let us consider the equivalent longitudinal stiffness of the truss beam. We find that the equivalent Young's modulus is

$$\tilde{E} = \frac{2EA}{h}\frac{1+\sqrt{2}\,E_0A_0/EA}{1+\frac{\sqrt{2}}{4}E_0A_0/EA}. \tag{2.2}$$

According to the theory of elastic bending, the bending stiffness of the equivalent elastic beam in Fig. 1 is then

$$\tilde{E}\tilde{I} = \int_{-h/2}^{-h/2} \tilde{E}y^2\mathrm{d}y = \frac{EAh^2}{6}\frac{1+\sqrt{2}\,E_0A_0/EA}{1+\frac{\sqrt{2}}{4}E_0A_0/EA}, \tag{2.3}$$

which depends on E_0A_0, and differs from (2.1). This means that when we regard the truss beam in Fig. 1 as a continuum, we are regarding it not as an elastic continuum but as some other continuum which admits bending stiffness independent from the elastic constants. This is equivalent to say that we are considering a continuum admitting an additional degree of freedom. In the following we shall define a continuum whose mechancal characteristics are in agreement with those implicitly assumed in the continuous treatment of trusses and then explore the analytical consequences on the basis of our theory.

Usually a large truss has a regular internal structure, and it can be regarded as an aggregate of "units" constructed by several members. If the unit is rectangular (or a parallelepiped in the three-dimensional case), it is deformed into a general quadrilateral (or a hexahedron). The deformation maps a line (or a plane) on a line (or a plane). Geometrically speaking, this fact characterizes *projective transformations* of the space. As is schematically shown in Fig. 2, if the displacement u^i of the center of the unit is fixed, a projective transformation is expressed as

$$x^{i'} = \frac{A^i_jx^j}{1+A_jx^j}, \tag{2.4}$$

when it does not differ very much from the identity mapping. In general, of course, we must use a homogeneous coordinate system, but we consider infinitesimal deformations only, so that it is sufficient to use an inhomogeneous coordinate system. We adopt the rule of

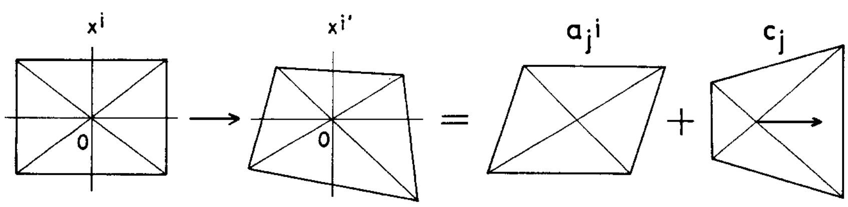

Figure 2. Schematic representation of an infinitesimal projective transformation.

summation convention of indices over 1, 2 or 1, 2, 3 throughout this paper, and the coordinate system is always Cartesian, so that there is no distinction between contravariant and covariant components of vectors and tensors. As is customary in tensor calculus, we use ( ) and [ ] to indicate the symmetric part and the antisymmetric part of components, respectively.

If deformations are small, we can put

$$A^i_j = \delta^i_j + a^i_j, \quad A_j = -\tfrac{1}{2}c_j, \tag{2.5}$$

where δ^i_j is Kronecker's delta, and a^i_j and c_j are small quantities. Then we can expand (2.4) with respect to a^i_j and c_j and we obtain

$$\delta x^i = a^i_j x^j + \tfrac{1}{2}c_j x^j x^i, \tag{2.6}$$

where $\delta x^i \equiv x^{i'} - x^i$, and terms of higher order are neglected. Then we can conclude that a^i_j represents the first-order deformations of the unit and that c_j represents expansions along the j-axis as is illustrated in Fig. 2. If another transformation of the same form

$$\delta x^j = b^i_j x^j + \tfrac{1}{2}d_j x^j x^i, \tag{2.7}$$

is applied after transformation (2.6), the composite is a transformation

$$\delta x^i = (a^i_j + b^i_j)x^j + \tfrac{1}{2}(c_j + d_j)x^j x^i \tag{2.8}$$

up to the second-order terms. This gives the *principle of superposition* of transformations of the form (2.6). In other words, transformations of the form (2.6) constitute a *commutative transformation group*.

The strain energy stored in a truss unit undergoing deformation (2.6) is a function of a^i_j and c_j. Let the energy divided by the area (or the volume) of the unit be $\varepsilon(a_{ji}, c_j)$. According to the *principle of material frame-indifference* [12], the energy form must be invariant to rigid rotations. If a rigid rotation of the form

$$\delta x^i = \omega_{ji} x^j, \tag{2.9}$$

where ω_{ji} is an antisymmetric tensor, is superposed on (2.6), the energy becomes $\varepsilon(a_{ji} + \omega_{ji}, c_j)$ due to the principle of superposition (2.8). Since

$$\varepsilon(a_{ji} + \omega_{ji}, c_j) = \varepsilon(a_{ji}, c_j) \tag{2.10}$$

must hold for an arbitrary antisymmetric tensor ω_{ji}, we can conclude that ε depends only on the symmetric part of a_{ji}

$$e_{ji} \equiv a_{(ji)} \tag{2.11}$$

and c_j. We call e_{ji} and c_j the *primary* and the *secondary* strains respectively. Stresses dual to these strain with respect the energy are defined as follows:

$$\delta\varepsilon = \sigma^{ji}\delta e_{ji} + \theta^j \delta c_j, \tag{2.12}$$

$$\sigma^{ji} \equiv \partial\varepsilon/\partial e_{ji}, \quad \theta^j \equiv \partial\varepsilon/\partial c_j. \tag{2.13}$$

Due to the symmetry of e_{ji} we can assign symmetry to σ^{ji}:

$$\sigma^{[ji]} = 0. \tag{2.14}$$

We call σ^{ji} and θ^j the *primary* stress and the *secondary* stress respectively. The primary stress σ^{ji} is the reaction against the primary strain e_{ji}, and the secondary stress θ^j is the reaction against the secondary strain c_j, or a quantity like the bending moment of the unit. Thus, taking the secondary stress into consideration, we can dissolve the inconsistency in the previous example of the truss beam.

3. Constitutive equations for a truss

Suppose an elastic rod is pin-jointed at two points $A(\xi^i)$ and $B(-\xi^i)$ as shown in Fig. 3a. The length of it is $2l = 2\|\xi^i\|$. Let the two points undergo small changes $\Delta_1\xi^i$ and $-\Delta_2\xi^i$ respectively. The length changes to $2l' = \|2\xi^i + \Delta_1\xi^i + \Delta_2\xi^i\|$, and the longitudinal strain is $(l' - l)/l = (1/2l^2)\xi^i(\Delta_1\xi^i + \Delta_2\xi^i)$. The stored strain energy is then

$$U_{AB} = \frac{EA}{4l^3}[\xi^i(\Delta_1\xi^i + \Delta_2\xi^i)]^2, \tag{3.1}$$

where EA is the longitudinal stiffness of the rod. Under our transformation (2.6), the increments $\Delta_1\xi^i$ and $\Delta_2\xi^i$ are

$$\Delta_1\xi^i = a^i_j\xi^j + \tfrac{1}{2}c_j\xi^j\xi^i, \quad \Delta_2\xi^i = a^i_j\xi^j - \tfrac{1}{2}c_j\xi^j\xi^i, \tag{3.2}$$

and hence

$$U_{AB} = \frac{EA}{l^3}(\xi^i\xi^j e_{ji})^2. \tag{3.3}$$

Next consider two rods of the same shape and of the same material pin-jointed as shown in Fig. 3b. The strain energy stored under transformation (2.6) is determined in the same

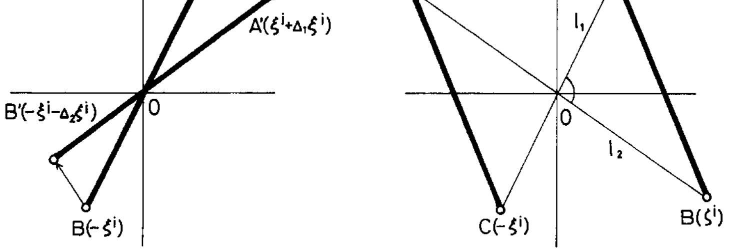

Figure 3. A truss unit is composed of (a) diagonal members and (b) circumference members.

manner. We finally obtain

$$U_{AB,CD} = \frac{EA}{\sqrt{(l_1^2 + l_2^2 - 2\alpha l_1 l_2)^3}} [\{(\xi^j\xi^i - 2\xi^j\zeta^i + \zeta^j\zeta^i)e_{ji}\}^2$$
$$+ \tfrac{1}{4}\{((l_1^2 - \alpha l_1 l_2)\xi^j + (l_2^2 - \alpha l_1 l_2)\zeta^j)c_j\}^2], \quad (3.4)$$

where $l_1 = \|\xi^i\|$, $l_2 = \|\zeta^i\|$, and α is the cosine of the angle determined by $OA$and OB.

In general the strain energy of a truss unit of arbitrary internal structure is a sum of the forms (3.3) and (3.4), and hence has the form

$$\varepsilon = \tfrac{1}{2}E^{lkji}e_{lk}e_{ji} + \tfrac{1}{2}F^{ji}c_jc_i. \quad (3.5)$$

Due to the symmetry of e_{ji}, we can assign the restrictions

$$E^{lkji} = E^{klji} = E^{lkij} = E^{jilk}, \quad F^{ji} = F^{ij}. \quad (3.6)$$

By definition the stresses are expressed as

$$\sigma^{lk} = E^{lkji}e_{ji}, \quad \theta^j = F^{ji}c_j, \quad (3.7)$$

which are the constitutive equations of the truss. We should note that the truss unit may be of arbitrary shape, and that E^{lkji} and F^{ji} may vary smoothly from unit to unit.

Consider a two-dimensional truss unit shown in Fig. 4, for example. The energy per unit area is written as

$$\varepsilon = \frac{1}{4\alpha\beta h^2}(U_{AC} + U_{BD} + \tfrac{1}{2}U_{AB,CD} + \tfrac{1}{2}U_{AD,BC}), \quad (3.8)$$

where the energy of the circumference members is divided by 2 because it is doubly counted when the sum over the whole area is taken. The constitutive equations are

$$\sigma^{xx} = \mu_1 e_{xx} + 2\lambda_2 e_{xy} + \lambda_1 e_{yy},$$
$$\sigma^{xy} = \lambda_2 e_{yy} + 2\lambda_1 e_{xy} + \lambda_3 e_{yy} \quad (= \sigma^{yx}), \quad (3.9)$$
$$\sigma^{yy} = \lambda_1 e_{xx} + 2\lambda_3 e_{xy} + \mu_2 e_{yy},$$
$$\theta^x = v_1 c_x, \quad \theta^y = v_2 c_y,$$

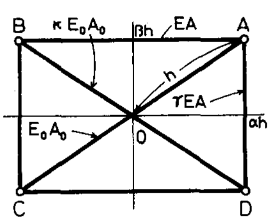

Figure 4. An example of a truss unit.

where

$$\mu_1 = \frac{\alpha^3(1+\kappa)}{2\alpha h} E_0 A_0 + \frac{1}{\beta h} EA, \quad \mu_2 = \frac{\beta^3(1+\kappa)}{2\alpha h} E_0 A_0 + \frac{\gamma}{\alpha h} EA,$$

$$\lambda_1 = \frac{\alpha\beta(1+\kappa)}{2h} E_0 A_0, \quad \lambda_2 = \frac{\alpha^2(1-\kappa)}{2h} E_0 A_0, \quad \lambda_3 = \frac{\beta^2(1-\kappa)}{2h} E_0 A_0,$$

$$\nu_1 = \frac{\alpha\gamma h}{4} EA, \quad \nu_2 = \frac{\beta h}{2} EA. \tag{3.10}$$

4. Strain-displacement relations and equations of equilibrium

In a truss one unit is pin-jointed to surrounding ones, so that each unit is not allowed independent deformations. Hence compatibility conditions must be required for a^i_j and c_j, if they are to be regarded as continuous field variables. Here we try to express a^i_j and c_j in terms of the displacement filed u^i of the center of the unit. As is shown in Fig. 5, the displacement of a point ξ^i is, if considered to be induced by deformation (2.6) of the unit, or the microscopic deformation, expressed in the form

$$\delta\xi^i = \xi^j a^i_j + \tfrac{1}{2}\xi^j\xi^i c_j. \tag{4.1}$$

If, on the other hand, it is considered to be induced by the displacement field u^i, or the macroscopic deformation, it can be expressed as the Taylor expansion

$$\delta u^i = \xi^j \partial_j u^i + \tfrac{1}{2}\xi^k\xi^j\partial_k\partial_j u^i \tag{4.2}$$

up to the second order. Throughout this paper ∂_j denotes $\partial/\partial x^j$. It is desirable that the microscopic deformation should coincide with the macroscopic deformation, i.e., the equality $\delta\xi^i = \delta u^i$ should hold for an arbitrary point ξ^i, but it is impossible because of the restricted form of our transformation (4.1). Let them coincide in the sense of the least-square error over some domain $\mathscr{D}$. This is equivalent to say that elastic springs are attached between the unit and the underlying field over the domain $\mathscr{D}$, for the potential energy of each spring is proportional to the square of the discrepancy there. Put

$$E = \int_{\mathscr{D}} \|(a^i_j - \partial_j u^i)\xi^j + \tfrac{1}{2}(c_j\delta^i_k - \partial_j\partial_k u^i)\xi^j\xi^k\|^2 \mathrm{d}\xi. \tag{4.3}$$

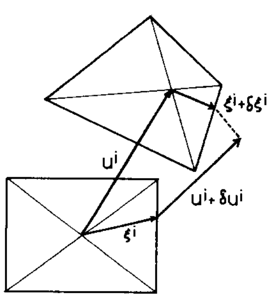

Figure 5. The discrepancy between the microscopic deformation and the macroscopic deformation.

If we take the domain $\mathscr{D}$ to be symmetric with respect to each axis of the coordinate, we obtain, from $\partial E/\partial a^i_j = 0$ and $\partial E/\partial c_j = 0$,

$$a^i_j = \partial_j u^i, \quad c_j = \frac{1}{m_j} T^{lkji}\partial_l\partial_k u^i, \tag{4.4}$$

where index j is not summed, and

$$m_j = \int_{\mathscr{D}} \|\xi^k\| (\xi^j)^2 \mathrm{d}\xi, \quad T^{lkji} = \int_{\mathscr{D}} \xi^l\xi^k\xi^j\xi^i \mathrm{d}\xi. \tag{4.5}$$

We can obtain an invariant form of c_j irrespective of the choice of the coordinate system, if we take $\mathscr{D}$ to be a spherical region:

$$c_j = \frac{1}{n+2}(\Delta u_j + 2\partial_j\partial_k u^k), \tag{4.6}$$

where Δ is the Laplacian operator, and n $(= 2, 3)$ is the dimensionality of the space. Generally $\mathscr{D}$ should be determined according to the geometry of the truss. Let us consider the two-dimensional truss unit in Fig. 4, for example. If $\mathscr{D}$ is taken to be the four points B, C and D, then we have

$$\begin{aligned} c_x &= \alpha^2\partial_x\partial_x u_x + \beta^2\partial_y\partial_y u_y + 2\beta^2\partial_x\partial_y u_y, \\ c_y &= \beta^2\partial_y\partial_y u_y + \alpha^2\partial_x\partial_x u_y + 2\alpha^2\partial_x\partial_y u_x. \end{aligned} \tag{4.7}$$

For the sake of simplicity let us use (4.6) in the following. Then the increment of the total energy is

$$\delta U = \int\left[\sigma^{ji}\partial_j\delta u_i + \frac{1}{n+2}(\theta^j\Delta\delta u_j + 2\theta^j\partial_j\partial_i\delta u_i)\right]\mathrm{d}V. \tag{4.8}$$

The virtual work done by external forces for the variation is

$$\delta W = \int b^i\delta u_i \mathrm{d}V + \int (t^i\delta u_i + s^i\partial_n\delta u_i)\mathrm{d}S, \tag{4.9}$$

where b^i is the equivalent body force per unit volume (or area), and t^i is the surface traction. The quantity s^i represents the reaction, like the bending moment, acting against the secondary strain through the boundary surface, and ∂_n denotes the differentiation along the surface unit normal n_i. According to the principle of virtual work, the equality $\delta U = \delta W$ must hold for an arbitrary variation δu_i. Integrating (4.8) by parts, and equating it with (4.9), we obtain the equation of equilibrium

$$\partial_j\sigma^{ji} + b^i = \frac{1}{n+2}(\Delta\theta^i + 2\partial_i\partial_j\theta^i) \tag{4.10}$$

with boundary conditions

$$n_j\sigma^{ji} = t^i + \frac{2}{n+2}[2n_{(j}\partial_{i)}\theta^j - n_jn_i\partial_n\theta^j$$
$$+ (\sum_{\alpha=1}^{n} n_jn_i/R_\alpha - \sum_{\alpha=1}^{n} l_j^{(\alpha)}l_i^{(\alpha)}/R_\alpha)\theta^j + \tfrac{1}{2}\partial_n\theta^i],$$
$$s^i = \frac{1}{n+2}(\theta^i + 2\theta^jn_jn_i), \tag{4.11}$$

where $l_j^{(\alpha)}$ is the α-th principal direction, and $1/R_\alpha$ is the corresponding principal curvature of the surface.

5. Displacement field in the three-dimensional space

Now we try to find a complete solution of equation (4.10) in the three-dimensional space. We assume the field to be isotropic. It can be said that the field corresponds to a truss of random connection. If isotropy is required, the constitutive equations (3.7) take the familiar form

$$\sigma^{ji} = 2\mu\left(e_{ji} + \frac{\sigma}{1-2\sigma}\delta_{ji}e_{kk}\right),$$
$$\theta^j = vc_j, \tag{5.1}$$

where μ and σ are material constants corresponding respectively to the shear modulus and Poisson's ratio of the linear elastic field, while v is a new material constant related to the microstructure of the field. Substituting (5.1) into (4.10) with $n = 3$, we obtain the displacement equation

$$\Delta(1 - l_0^2\Delta)\boldsymbol{u} + (\alpha - 8l_0^2\Delta)\nabla\nabla\cdot\boldsymbol{u} + \frac{1}{\mu}\boldsymbol{b} = 0, \tag{5.2}$$

where we have put $\alpha = 1/(1-2\sigma)$ and $l_0 = (1/5)\sqrt{v/\mu}$. The new constant l_0 has the dimension of length and is about the order of the scale of microstructure. In the limit of $l_0 = 0$, equation (5.2) reduces to that of elasticity. We follow the procedure of Mindlin and Tiersten [13] to obtain a complete solution of the displacement equation. Substitution of the resolution

$$\boldsymbol{u} = \nabla\phi + \nabla\times\boldsymbol{H} \tag{5.3}$$

into (5.2) yields

$$\Delta[(1 + \alpha - 9l_0^2\Delta)\nabla\phi + (1 - l_0^2\Delta)\nabla\times\boldsymbol{H}] = -\frac{1}{\mu}\boldsymbol{b}. \tag{5.4}$$

We can put

$$(1 + \alpha - 9l_0^2\Delta)\nabla\phi + (1 - l_0^2\Delta)\nabla \times \boldsymbol{H} = \boldsymbol{B}, \tag{5.5}$$

$$\Delta\boldsymbol{B} = -\frac{1}{\mu}\boldsymbol{b}. \tag{5.6}$$

Taking the divergence of (5.5), we obtain

$$\Delta(1 - l_1^2\Delta)\phi = \frac{1}{1+\alpha}\nabla\cdot\boldsymbol{B}, \tag{5.7}$$

where we have put $l_1 = 3l_0\sqrt{1+\alpha}$. Integration of (5.7) yields

$$(1 - l_1^2\Delta)\phi = \frac{1}{2(1+\alpha)}(\boldsymbol{r}\cdot\boldsymbol{B} + B_0), \tag{5.8}$$

$$\Delta B_0 = \frac{1}{\mu}\boldsymbol{r}\cdot\boldsymbol{b}. \tag{5.9}$$

Again (5.8) is integrated to give ϕ in the form

$$\phi = \frac{1}{2(1+\alpha)}(\boldsymbol{r}\cdot\boldsymbol{B} + B_0) + \frac{l_1^2}{1+\alpha}\nabla\cdot\boldsymbol{B} + B_1, \tag{5.10}$$

$$(1 - l_1^2\Delta)B_1 = -\frac{l_1^4}{\mu(1+\alpha)}\nabla\cdot\boldsymbol{b}. \tag{5.11}$$

From (5.5) we have

$$(1 - l_0^2\Delta)\nabla \times \boldsymbol{H} = \boldsymbol{B} - \tfrac{1}{2}\nabla(\boldsymbol{r}\cdot\boldsymbol{B} + B_0). \tag{5.12}$$

Integration of this yields

$$\nabla \times \boldsymbol{H} = \boldsymbol{B} - \tfrac{1}{2}\nabla(\boldsymbol{r}\cdot\boldsymbol{B} + B) - l_0^2\nabla\nabla\cdot\boldsymbol{B} + \boldsymbol{B}', \tag{5.13}$$

$$(1 - l_0^2\Delta)\boldsymbol{B}' = -\frac{l_0^2}{\mu}\boldsymbol{b} + \frac{l_0^4}{\mu}\nabla\nabla\cdot\boldsymbol{b}. \tag{5.14}$$

From (5.3), (5.10) and (5.13) we finally obtain

$$\boldsymbol{u} = \boldsymbol{B} - \tilde{\alpha}\nabla(\boldsymbol{r}\cdot\boldsymbol{B} + B_0) - \tilde{\beta}\nabla\nabla\cdot\boldsymbol{B} + \boldsymbol{B}' + \nabla B_1, \tag{5.15}$$

$$\begin{aligned} &\mu\Delta\boldsymbol{B} = -\boldsymbol{b}, \quad \mu(1 - l_0^2\Delta)\boldsymbol{B}' = -l_0^2\boldsymbol{b} + l_0^4\nabla\nabla\cdot\boldsymbol{b}, \\ &\mu\Delta B_0 = \boldsymbol{r}\cdot\boldsymbol{b}, \quad \mu(1 - l_0^2\Delta)B_1 = -\alpha' l_1^4\nabla\cdot\boldsymbol{b}, \end{aligned} \tag{5.16}$$

where

$$\tilde{\alpha} = \frac{1}{4(1-\sigma)}, \quad \tilde{\beta} = -\frac{l_0^2(1-4\sigma)(5-8\sigma)}{4(1-\sigma)^2}, \quad \alpha' = \frac{1-2\sigma}{2(1-\sigma)}. \tag{5.17}$$

In the limit of $l_0 \to 0$, the functions $\boldsymbol{B}'$ and B_1 vanish, and $\boldsymbol{B}$ and B_0 reduce to the Papkovitch functions of linear elasticity.

If the body force $\boldsymbol{b}$ is absent, we have from (4.6) and (5.1)

$$\boldsymbol{\Theta} = 5\mu l_0^2 \left[\frac{1-4\sigma}{2(1-\sigma)} \nabla\nabla\cdot\boldsymbol{B} + 2\nabla\nabla\cdot\boldsymbol{B}' + \frac{1}{l_0^2}\boldsymbol{B}' + \frac{3}{l_1^2}\nabla B_1 \right]. \tag{5.18}$$

Assume the elastic solution $\boldsymbol{u} = \boldsymbol{B} - \tilde{\alpha}\nabla(\boldsymbol{r}\cdot\boldsymbol{B} + B_0)$. Then the secondary stress is

$$\boldsymbol{\Theta} = 5\mu l_0^2 \frac{1-4\sigma}{2(1-\sigma)} \nabla\nabla\cdot\boldsymbol{B}, \tag{5.19}$$

which vanishes when $\sigma = \frac{1}{4}$. This fact implies that the microscopic deformation of the elastic field is nearly the first-order deformation only when $\sigma = \frac{1}{4}$. In this case the elastic field is also the solution of the problem (4.10) and (4.11).

In an infinite field equations (5.16) are integrated to yield

$$\begin{aligned} \boldsymbol{B} &= \frac{1}{4\pi\mu}\int \frac{\boldsymbol{b}}{r}\mathrm{d}V, \quad \boldsymbol{B}' = \frac{1}{4\pi\mu l_0}\int \psi\left(\frac{r}{l_0}\right)(\boldsymbol{b} - l_0^2\nabla\nabla\cdot\boldsymbol{b})\mathrm{d}V, \\ B_0 &= -\frac{1}{4\pi\mu}\int \frac{\boldsymbol{r}\cdot\boldsymbol{b}}{r}\mathrm{d}V, \quad B_1 = \frac{\alpha' l_1}{4\pi\mu}\int \psi\left(\frac{r}{l_1}\right)\nabla\cdot\boldsymbol{b}\mathrm{d}V, \end{aligned} \tag{5.20}$$

where $\psi(r) = -e^{-r}/r$, and $r = \|\boldsymbol{r}\|$. Consider the concentrated body force at the origin, and take the limit

$$\lim_{V\to 0}\int \boldsymbol{b}\mathrm{d}V = \boldsymbol{F}. \tag{5.21}$$

Then expressions (5.20) reduce to

$$\begin{aligned} B_i &= \frac{1}{4\pi\mu r}F_i, \quad B_i' = \frac{1}{4\pi\mu}\left[\left(\frac{\psi_0}{l_0} - \frac{\psi_0'}{r}\right)\delta^{ji} + \left(\frac{\psi_0'}{r^3} - \frac{\psi_0''}{l_0 r^2}\right)x^j x^i\right]F_j, \\ B_0 &= 0, \quad B_1 = -\frac{\alpha'}{4\pi\mu}\frac{\psi_1'}{r}x^i F_i, \end{aligned} \tag{5.22}$$

where $\psi_0 = \psi(r/l_0)$, $\psi_0' = \psi'(r/l_0)$, $\psi_0'' = \psi''(r/l_0)$, $\psi_1' = \psi'(r/l_1)$ and $\psi'(\cdot)$ denotes the derivation with respect to the argument. Employing (5.22) in (5.15), we obtain

$$u_i = \frac{1}{4\pi\mu}\Phi_{ji}F^j, \tag{5.23}$$

$$\Phi_{ji}(\boldsymbol{r}) = \left[\frac{3-4\sigma}{4(1-\sigma)}\frac{1}{r} - \frac{(1-4\sigma)(5-8\sigma)}{4(1-\sigma)^2}\frac{l_0^2}{r^3} - \frac{1}{l_0}\psi_1\left(\frac{r}{l_0}\right)\right.$$
$$\left. - \frac{1-2\sigma}{2(1-\sigma)}\frac{1}{l_1}\psi_2\left(\frac{r}{l_1}\right)\right]\delta_{ji} + \left[\frac{1}{4(1-\sigma)}\frac{1}{r^3} + \frac{3(1-\sigma)(5-8\sigma)}{4(1-\sigma)^2}\frac{l_0^2}{r^3}\right.$$
$$\left. + \frac{1}{l_0^3}\psi_3\left(\frac{r}{l_0}\right) + \frac{1-2\sigma}{2(1-\sigma)}\frac{1}{l_1}\psi_3\left(\frac{r}{l_1}\right)\right]x^j x^i, \tag{5.24}$$

$$\psi_1(r) = \left(\frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^3}\right)e^{-r}, \quad \psi_2(r) = \left(\frac{1}{r^2} + \frac{1}{r^3}\right)e^{-r},$$
$$\psi_3(r) = \left(\frac{1}{r^3} + \frac{3}{r^4} + \frac{3}{r^5}\right)e^{-r}. \tag{5.25}$$

Then we can obtain the displacement field with distributed body force by superposition.

$$u_i(\boldsymbol{r}) = \int \Phi_{ji}(\boldsymbol{r} - \boldsymbol{r}')b^j(\boldsymbol{r}')\mathrm{d}\boldsymbol{r}'. \tag{5.26}$$

6. Cylindrical and spherical cavities in a field of simple tension

Consider boundary-value problems of equation (5.2) with the body force absent. Equation (5.2) is regarded as a singular perturbation to the displacement equation of elasticity. As is well known in the theory of singular perturbation [14], the unperturbed solution is in fairly good agreement with the perturbed solution except in the "boundary layer" whose thickness is about l_0. We now study the surface singularity by investigating the stresses around cylindrical and spherical cavities in a field of simple tension.

Consider a cylindrical cavity of radius a of infinite length whose axis coincides with the z-axis. A simple tension τ in the x-direction is given in cylindrical coordinates r, θ, z by

$$\sigma^{rr} = \frac{\tau}{2}(1 + \cos 2\theta), \quad \sigma^{\theta\theta} = \frac{\tau}{2}(1 - \cos 2\theta), \quad \sigma^{r\theta} = -\frac{\tau}{2}\sin 2\theta, \tag{6.1}$$

and the remaining components are zero. We wish to add a stress field which will produce a free surface at $r = a$ and vanishes at infinity. We take the potential functions, for the additional field, to be of the form

$$B_x = B(r,\theta), \quad B_y = B_z = 0, \quad B_0 = B_0(r,\theta),$$
$$B' = B'(r,\theta), \quad B'_y = B'_z = 0, \quad B_1 = B_1(r,\theta), \tag{6.2}$$

$$\Delta B = 0, \quad (1 - l_0^2\Delta)B' = 0,$$
$$\Delta B_0 = 0, \quad (1 - l_1^2\Delta)B_1 = 0. \tag{6.3}$$

It can be shown that the desired potential functions have the form

$$B = \frac{A_1}{r}\cos\theta,\quad B' = A_2K_1\left(\frac{r}{l_0}\right) + A_3K_2\left(\frac{r}{l_1}\right)\cos 2\theta,$$
$$B_0 = A_4\log r + \frac{A_3}{r^2}\cos 2\theta,\quad B_1 = A_5K_0\left(\frac{r}{l_1}\right) + A_6K_2\left(\frac{r}{l_1}\right)\cos 2\theta, \tag{6.4}$$

where K_n is the n-th modified Bessel function of the second kind. The constants $A_1 \sim A_6$ are determined by the condition at $r = a$

$$\theta_r = 0,\quad \theta_\theta = 0,$$
$$\sigma^{rr} - \eta_r = -\frac{\tau}{2}(1 + \cos 2\theta),\quad \sigma^{r\theta} - \eta_\theta = \frac{\tau}{2}\sin 2\theta, \tag{6.5}$$

where

$$\eta_r = \frac{1}{5}\left(\frac{3}{2}\partial_r\theta_r + \frac{\theta_r}{r} + \frac{1}{r}\partial_\theta\theta_\theta\right),$$
$$\eta_\theta = \frac{1}{5}\left(\frac{1}{r}\partial_\theta\theta_r + \frac{1}{2}\partial_r\theta_\theta - \frac{\theta_\theta}{r}\right). \tag{6.6}$$

Explicit expressions for $A_1 \sim A_6$ are tedious and difficult to obtain, so that we will resort to numerical calculations using a computer. The maximum tension at the surface arises at $\theta = \pm(\pi/2)$. (See Fig. 6a.) As was predicted in the previous section, the stress coincides with that of elasticity when $\sigma = \frac{1}{4}$. The maximum stress is below the elastic limit when $\sigma > \frac{1}{4}$, while it is intensified when $\sigma < \frac{1}{4}$. This is attributed to the different profiles of the secondary stress near the surface, which are shown in Fig. 6b.

Stresses around a spherical cavity are obtained in the same manner. Let the cavity be a

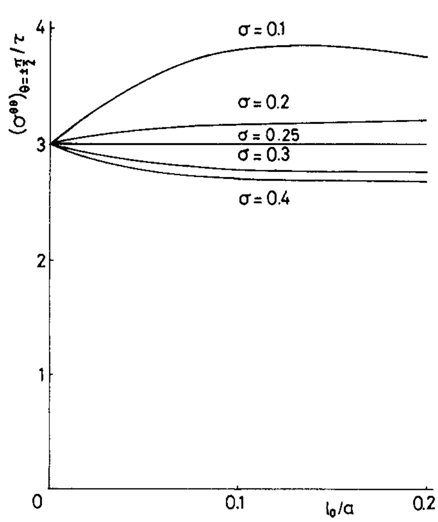

Figure 6a. The maximum stress $\sigma^{\theta\theta}$ at the surface of a cylindrical cavity in a field of simple tension τ.

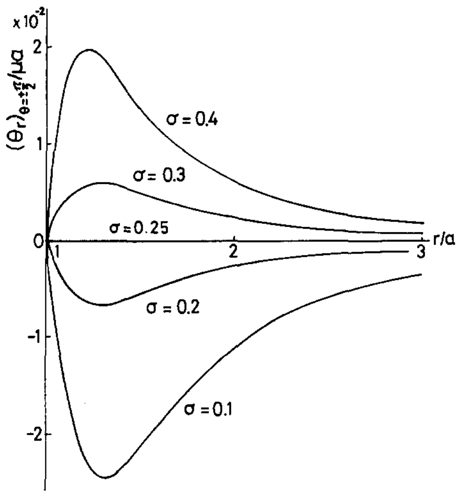

Figure 6b. The secondary stress θ_r near the surface of a cylindrical cavity. $\theta = \pm(\pi/2)$. $l_0/a = 0.1$.

sphere of radius a whose center is at the origin and the tension τ be in the z-direction. We seek a field to be added to the potential functions of the form

$$\begin{aligned} &B_x = B_y = 0, \quad B_z = B(r, \theta), \quad B_0 = B_0(r, \theta), \\ &B'_x = B'_y = 0, \quad B'_z = B'(r, \theta), \quad B_1 = B_1(r, \theta), \end{aligned} \tag{6.6}$$

in spherical coordinates r, θ, ϕ. These functions are determined in the following form:

$$\begin{aligned} &B = \frac{A_1}{r^2}\cos\theta. \quad B_0 = \frac{A_2}{r} + \frac{A_3}{r^3}(1 + 3\cos 2\theta), \\ &B' = A_4\left(\frac{l_0}{r} + \left(\frac{l_0}{r}\right)^2\right)e^{-r/l_0}\cos\theta, \\ &B_1 = A_5\frac{l_1}{r}e^{-r/l_1} + A_6\left(\frac{l_1}{r} + 3\left(\frac{l_1}{r}\right)^2 + 3\left(\frac{l_1}{r}\right)^3\right)e^{-r/l_1}(1 + 3\cos 2\theta). \end{aligned} \tag{6.7}$$

The indeterminate coefficients $A_1 \sim A_6$ are determined by boundary conditions (6.5), where instead of (6.6),

$$\begin{aligned} &\eta_r = \frac{1}{5}\left(\frac{3}{2}\partial_r\theta_r + \frac{2}{r}\theta_r + \frac{1}{r}\left(\partial_\theta\theta_\theta + \frac{1}{\tan\theta}\right)\right), \\ &\eta_\theta = \frac{1}{5}\left(\frac{1}{r}\partial_\theta\theta_r + \frac{1}{2}\partial_r\theta_\theta - \frac{\theta_\theta}{r}\right). \end{aligned} \tag{6.8}$$

The maximum tension at the surface arises again at $\theta = \pm(\pi/2)$. (See Fig. 7a.) The behavior is similar to the previous case, but, as is expected, deviations from the elastic limit are not so remarkable. Fig. 7b shows the profile of the secondary stress near the surface.

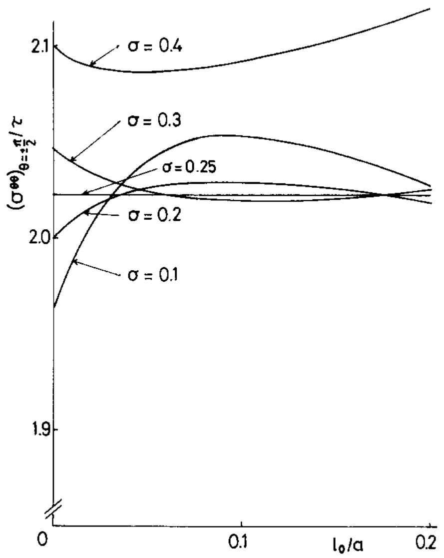

Figure 7a. The maximum stress $\sigma^{\theta\theta}$ at the surface of a spherical cavity in a field of simple tension τ.

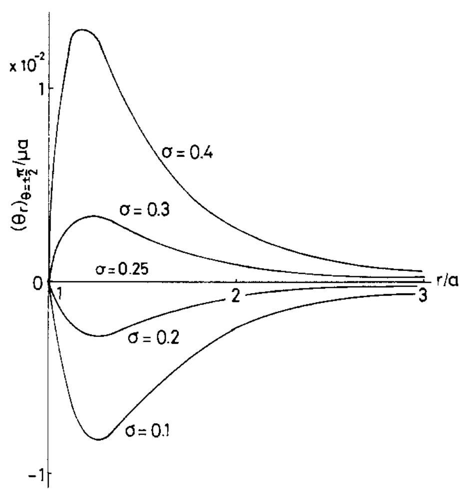

Figure 7b. The secondary stress θ_r near the surface of a spherical cavity. $\theta = \pm(\pi/2)$. $l_0/a = 0.1$.

7. Conclusion

We have described a large truss by a continuum which admits projective transformations as microscopic deformations. A suitable set of parameters which characterizes the microscopic deformations is extracted, and its spatial compatibility is studied. The equation of equilibrium is derived by means of a variational principle. Following the procedure of Mindlin and Tiersten [13], we have exhibited a complete solution in the three-dimensional isotropic space. Finally, stresses around cylindrical and spherical cavities in a field of simple tension are analyzed to see the singular effects near the boundary surface.

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